

ON EXACT ∞ -CATEGORIES AND THE THEOREM OF THE HEART

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ABSTRACT. The homotopy theory of exact ∞ -categories is introduced and employed to prove Amnon Neeman’s Theorem of the Heart for Waldhausen K -theory. This implies a new compatibility between Waldhausen K -theory and Neeman K -theory, and it provides a new proof of the Dévissage and Localization Theorems of Blumberg–Mandell.

0. INTRODUCTION

Over the course of the past 20 years, Amnon Neeman has advanced the *algebraic K -theory of triangulated categories* [9, 10, 11, 12, 13, 14, 15, 16, 17] as a way of extracting K -theoretic data directly from the triangulated homotopy category of a stable homotopy theory. This form of K -theory has known limitations: an example of Marco Schlichting [18] shows that Waldhausen K -theory can distinguish stable ∞ -categories with equivalent triangulated homotopy categories. Nevertheless, one of the most promising insights from the study of the algebraic K -theory of triangulated categories is Neeman’s *Theorem of the Heart* [12, 13, 16], which expresses an equivalence between the Neeman K -theory of a triangulated category \mathcal{T} equipped with a bounded t -structure and the Quillen K -theory of its heart \mathcal{T}^\heartsuit . The main result of this paper, Th. 5.6, is the natural analogue of Neeman’s Theorem of the Heart for the *Waldhausen K -theory of ∞ -categories* developed in [3].

More precisely, this paper introduces a natural homotopy-theoretic generalization of Quillen’s notion of an exact category — that of an *exact ∞ -category* (Df. 2.4). This notion, because it requires a compatibility between certain homotopy limits and certain homotopy colimits, is difficult to express in the more classical language of relative categories or ordinary Waldhausen categories. Exact ∞ -categories form a full subcategory of the ∞ -category of Waldhausen ∞ -categories (Cor. 3.4.1), and when the Waldhausen K -theory functor constructed in [3] is restricted to exact ∞ -categories, it enjoys a *self-duality* (Cor. 4.17.1). This self-duality is used to prove the following analogue of Neeman’s Theorem of the Heart.

Theorem (Heart, 5.6). *The Waldhausen K -theory of any idempotent-complete stable ∞ -category equipped with a bounded t -structure agrees with the Quillen K -theory of its heart.*

Whereas Neeman’s proof is notoriously lengthy and difficult [9, pp. 347–353], the proof of Th. 5.6 is brief, conceptual, and logically independent of Neeman’s. In fact, in the context of exact ∞ -categories, the proof is an amalgam of self-duality, a form of Waldhausen’s Fibration Theorem, and the Eilenberg Swindle. We therefore regard this theorem and its proof as a conclusive answer to [17, Problem 76].

We emphasize that the full strength of the conceptual apparatus introduced here and in [3] appears to be strictly necessary for a proof of this kind to work. In view of [18, Pr. 2.2], the use of the Fibration Theorem makes it unlikely that a proof along these lines could be adapted to the context of triangulated categories. Additionally, a form of K -theory that only

applies to stable ∞ -categories (or even to exact ∞ -categories in which every morphism is ingressive) would be inadequate even to *express* the relevant cases of self-duality.

Our main result also yields a comparison between Neeman's K -theory and Waldhausen K -theory in many cases. Indeed, the conjunction of Neeman's Theorem of the Heart and Th. 5.6 implies that the Waldhausen K -theory of an idempotent-complete stable ∞ -category \mathcal{A} agrees with the Neeman K -theory of its triangulated homotopy category $\mathcal{T} = h\mathcal{A}$ (the variant denoted $K({}^w\mathcal{T})$ in [17]), whenever the latter admits a bounded t-structure (Cor. 5.6.1). This verifies a conjecture of Neeman [13, Conj. A.5] for such stable homotopy theories.

This paper ends with a discussion of some immediate corollaries of the main theorem, which is employed

- to give a new, short proof of the theorem of Gillet, Thomason, and Waldhausen (Pr. 6.1) that compares the K -theory of a Karoubian exact category to the K -theory of its bounded derivade category,
- to give a new, short proof of the Dévissage and Localization Theorems of Blumberg–Mandell [4] (Pr. 7.3 and Th. 7.7), which yields a host of useful fiber sequences in the algebraic K -theory of ring spectra (Ex. 7.8), and
- to show that the G -theory of Deligne–Mumford stacks is invariant under derived thickenings (Pr. 8.2).

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1. TRIPLES OF ∞ -CATEGORIES

In [3], we defined *Waldhausen ∞ -categories* as certain *pairs* consisting of an ∞ -category and a subcategory thereof. Here, we shall define *exact ∞ -categories* as certain *triples* consisting of an ∞ -category and two subcategories thereof. It is therefore necessary for us to study the homotopy theory of these triples.

1.1. Definition. (1.1.1) A *triple* $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ of ∞ -categories (or simply a *triple*) is an ∞ -category \mathcal{C} , a pair structure $(\mathcal{C}, \mathcal{C}_\dagger)$ on \mathcal{C} , and a pair structure $(\mathcal{C}, \mathcal{C}^\dagger)$ on \mathcal{C}^{op} . As in [3, 1.19], the morphisms of \mathcal{C}_\dagger will be called *ingressive morphisms* or *cofibrations*, whereas the morphisms of \mathcal{C}^\dagger will be called *egressive morphisms* or *fibrations*.

(1.1.2) A *functor of triples* $\mathcal{C} \rightarrow \mathcal{D}$ consists of two functors of pairs,

$$(\mathcal{C}, \mathcal{C}_\dagger) \rightarrow (\mathcal{D}, \mathcal{D}_\dagger) \quad \text{and} \quad (\mathcal{C}, \mathcal{C}^\dagger) \rightarrow (\mathcal{D}, \mathcal{D}^\dagger),$$

with the same underlying functor $\mathcal{C} \rightarrow \mathcal{D}$; i.e., a functor of triples is a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_\dagger & \longrightarrow & \mathcal{D}_\dagger \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{D} \\ \uparrow & & \uparrow \\ \mathcal{C}^\dagger & \longrightarrow & \mathcal{D}^\dagger \end{array}$$

of ∞ -categories.

- (1.1.3) A functor of triples $\mathcal{C} \rightarrow \mathcal{D}$ is said to be *strict* if the induced functors of pairs $(\mathcal{C}, \mathcal{C}_\dagger) \rightarrow (\mathcal{D}, \mathcal{D}_\dagger)$ and $(\mathcal{C}, \mathcal{C}^\dagger) \rightarrow (\mathcal{D}, \mathcal{D}^\dagger)$ are each strict, i.e., if the diagrams

$$\begin{array}{ccc} \mathcal{C}_\dagger & \longrightarrow & \mathcal{D}_\dagger \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{D} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{C}^\dagger & \longrightarrow & \mathcal{D}^\dagger \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{D} \end{array}$$

are each pullback diagrams. In this case, we will say that the triple structure on \mathcal{C} is *induced* by the one on \mathcal{D} .

- 1.2. **Notation.** (1.2.1) For any two triples \mathcal{C} and \mathcal{D} , we denote by $\text{Fun}_{\mathbf{Trip}_\infty}^\#(\mathcal{C}, \mathcal{D})$ the ∞ -category of morphisms of pairs defined by the formula

$$\begin{aligned} \text{Fun}_{\mathbf{Trip}_\infty}^\#(\mathcal{C}, \mathcal{D}) &:= \text{Fun}_{\mathbf{Pair}_\infty}^\#(\mathcal{C}, \mathcal{D}) \times_{\text{Fun}(\mathcal{C}, \mathcal{D})} \text{Fun}_{\mathbf{Pair}_\infty}^\#(\mathcal{C}^{\text{op}}, \mathcal{D}^{\text{op}}) \\ &\cong \text{Fun}(\mathcal{C}^\dagger, \mathcal{D}^\dagger) \times_{\text{Fun}(\mathcal{C}^\dagger, \mathcal{D})} \text{Fun}(\mathcal{C}, \mathcal{D}) \times_{\text{Fun}(\mathcal{C}_\dagger, \mathcal{D})} \text{Fun}(\mathcal{C}_\dagger, \mathcal{D}_\dagger). \end{aligned}$$

- (1.2.2) For any two triples \mathcal{C} and \mathcal{D} , we denote by $\text{Fun}_{\mathbf{Trip}}^b(\mathcal{C}, \mathcal{D})$ the full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors that carry ingressesives to ingressesives and egressives to egressives.
- (1.2.3) Write $\mathbf{Trip}_\infty^\Delta$ for the fibrant simplicial category whose objects are triples and in which, for any triples \mathcal{C} and \mathcal{D} , the space of morphisms from \mathcal{C} to \mathcal{D} is given by

$$\mathbf{Trip}_\infty^\Delta(\mathcal{C}, \mathcal{D}) := \iota \text{Fun}_{\mathbf{Trip}_\infty}^\#(\mathcal{C}, \mathcal{D}).$$

- 1.3. Note that the simplicial nerve \mathbf{Trip}_∞ of $\mathbf{Trip}_\infty^\Delta$ can equivalently be described as a pullback:

$$\mathbf{Trip}_\infty := \mathbf{Pair}_\infty \times_{\mathbf{Cat}_\infty} \mathbf{Pair}_\infty.$$

As with the category of pairs [3, Pr. 1.18], we have the following.

- 1.4. **Proposition.** Denote by $w\mathbf{Trip}_\infty^0 \subset \mathbf{Trip}_\infty^0$ the subcategory consisting of those functors of pairs $\mathcal{C} \rightarrow \mathcal{D}$ whose underlying functor of ∞ -categories is a categorical equivalence that induces equivalences $h\mathcal{C}_\dagger \simeq h\mathcal{D}_\dagger$ and $h\mathcal{C}^\dagger \simeq h\mathcal{D}^\dagger$. Then \mathbf{Trip}_∞ is a relative nerve of $(\mathbf{Trip}_\infty^0, w\mathbf{Trip}_\infty^0)$.

- 1.5. **Definition.** Suppose $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ a triple. Then the *opposite triple* is the triple $\mathcal{C}^{\text{op}} := (\mathcal{C}^{\text{op}}, \mathcal{C}_\dagger^{\text{op}}, \mathcal{C}_\dagger^{\text{op}})$.

- 1.6. **Lemma.** The formation of the opposite pair defines an involution $(-)^{\text{op}}$ of the ∞ -category \mathbf{Trip}_∞ .

- 1.7. **Example.** Any ∞ -category C can be given the structure of a pair in two ways: the *minimal pair* $C^\flat := (C, \iota C)$ and the *maximal pair* $C^\sharp := (C, C)$. Any pair $\mathcal{C} = (\mathcal{C}, \mathcal{C}_\dagger)$ can be viewed as a triple in two ways: the *minimal triple* $\mathcal{C}^\flat := (\mathcal{C}, \mathcal{C}_\dagger, \iota \mathcal{C})$ and the *maximal triple* $\mathcal{C}^\sharp := (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C})$.

2. EXACT ∞ -CATEGORIES

Suppose C an ∞ -category that admits direct sums. Then the homotopy category hC is easily seen to admit direct sums as well. Moreover, the mapping spaces in C admit the natural structure of a homotopy-commutative H -space: for any morphisms $f, g \in \text{Map}_C(X, Y)$, one may define $f + g \in \text{Map}_C(X, Y)$ as the composite

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y.$$

2.1. Definition. An ∞ -category C will be said to be *additive* if its homotopy category hC is additive.

2.2. An ∞ -category C with direct sums is additive just in case, for any objects X and Y , the *shear map*

$$\text{Map}_C(X, Y) \times \text{Map}_C(X, Y) \longrightarrow \text{Map}_C(X, Y) \times \text{Map}_C(X, Y)$$

in the homotopy category of Kan simplicial sets given informally by $(f, g) \mapsto (f, f + g)$ is an isomorphism. Note in particular that additivity is a condition, not additional structure.

2.3. Example. Clearly the nerve of any ordinary additive category is an additive ∞ -category. Similarly, any stable ∞ -category is additive.

2.4. Definition. (2.4.1) Suppose $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ a triple of ∞ -categories. A pullback square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y', \end{array}$$

is said to be *ambigressive* if $X' \rightarrow Y'$ is ingressive and $Y \rightarrow Y'$ is egressive. Dually, a pushout square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y', \end{array}$$

is said to be *ambigressive* if $X \rightarrow Y$ is ingressive and $X \rightarrow X'$ is egressive.

(2.4.2) An *exact ∞ -category* $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is a triple of ∞ -categories satisfying the following conditions.

(2.4.2.1) The underlying ∞ -category \mathcal{C} is additive.

(2.4.2.2) The pair $(\mathcal{C}, \mathcal{C}_\dagger)$ is a Waldhausen ∞ -category.

(2.4.2.3) The pair $(\mathcal{C}, \mathcal{C}^\dagger)$ is a coWaldhausen ∞ -category.

(2.4.2.4) A square in \mathcal{C} is an ambigressive pullback if and only if it is an ambigressive pushout.

(2.4.3) In an exact ∞ -category, an *exact sequence* is an ambigressive pushout/pullback square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X''; \end{array}$$

we will abuse notation by writing

$$X' \twoheadrightarrow X \rightarrow X''$$

for this square. The cofibration $X' \twoheadrightarrow X$ will be called the *fiber* of $X \rightarrow X''$, and the fibration $X \rightarrow X''$ will be called the *cofiber* of $X' \twoheadrightarrow X$.

2.5. Thomason and Trobaugh called a suitable triple (C, C_+, C^\dagger) of ordinary categories whose nerves satisfy conditions (2.4.2.2-3) a *category with bifibrations*. When a class of weak equivalences is included, Thomason and Trobaugh used the term *biWaldhausen category*. This notion does not require additivity or any relationship between ambigressive pushout squares and ambigressive pullback squares; however, the relative nerve of a *complicial biWaldhausen category* can be seen to admit the structure of an exact ∞ -category.

2.6. Note that in an exact ∞ -category, a morphism of an exact ∞ -category is egressive just in case it appears as the cofiber of an ingressive morphism, and, dually, a morphism of an exact ∞ -category is ingressive just in case it appears as the fiber of an egressive morphism. Indeed, any cofiber of an ingressive morphism is egressive, and any egressive morphism is equivalent to the cofiber of its fiber. This proves the first statement; the second is dual. Consequently, the class of cofibrations in an exact ∞ -category specifies the class of fibrations, and vice versa.

2.7. **Example.** (2.7.1) The nerve NC of an ordinary category C can be endowed with a triple structure yielding an exact ∞ -category if and only if C is an ordinary exact category, in the sense of Quillen, wherein the admissible monomorphisms are exactly the cofibrations, and the admissible epimorphisms are exactly the fibrations. This is proved by appealing to the “minimal” axioms of Keller [5, App. A].

(2.7.2) At the other extreme, any stable ∞ -category is an exact ∞ -category in which all morphisms are both egressive and ingressive, and, conversely, any ∞ -category that can be regarded as an exact category with the *maximal* triple structure (in which any morphism is both ingressive and egressive) is a stable ∞ -category.

2.8. **Example.** We may interpolate between these two extremes. Suppose \mathcal{A} a stable ∞ -category equipped with a t-structure, and suppose $a, b \in \mathbf{Z}$.

(2.8.1) The ∞ -category $\mathcal{A}_{[a, +\infty)} := \mathcal{A}_{\geq a}$ admits an exact ∞ -category structure, in which every morphism is ingressive, but a morphism $Y \rightarrow Z$ is egressive just in case the induced morphism $\pi_a X \rightarrow \pi_a Y$ is an epimorphism of \mathcal{A}^\heartsuit .

(2.8.2) Dually, the ∞ -category $\mathcal{A}_{(-\infty, b]} := \mathcal{A}_{\leq b}$ admits an exact ∞ -category structure, in which every morphism is egressive, but a morphism $X \rightarrow Y$ is ingressive just in case the induced morphism $\pi_b X \rightarrow \pi_b Y$ is a monomorphism of \mathcal{A}^\heartsuit .

(2.8.3) We may intersect these subcategories to obtain the full subcategory

$$\mathcal{A}_{[a, b]} := \mathcal{A}_{\geq a} \cap \mathcal{A}_{\leq b} \subset \mathcal{A},$$

and we may intersect the subcategories of ingressive and egressive morphisms described to obtain the following exact ∞ -category structure on $\mathcal{A}_{[a, b]}$. A morphism $X \rightarrow Y$ is ingressive just in case the induced morphism $\pi_b X \rightarrow \pi_b Y$ is a monomorphism of the abelian category \mathcal{A}^\heartsuit . A morphism $Y \rightarrow Z$ is egressive just in case the induced morphism $\pi_a X \rightarrow \pi_a Y$ is an epimorphism of \mathcal{A}^\heartsuit .

2.9. **Example.** Yet more generally, suppose \mathcal{A} a stable ∞ -category, and suppose $\mathcal{C} \subset \mathcal{A}$ any full additive subcategory that is closed under extensions. Declare a morphism $X \rightarrow Y$ of \mathcal{C} to be ingressive just in case its cofiber in \mathcal{A} lies in \mathcal{C} . Dually, declare a morphism $Y \rightarrow Z$ of \mathcal{C} to be egressive just in case its fiber in \mathcal{A} lies in \mathcal{C} . Then \mathcal{C} is exact with this triple structure.

2.10. **Definition.** Suppose \mathcal{C} and \mathcal{D} two exact ∞ -categories. A functor of triples $\mathcal{C} \rightarrow \mathcal{D}$ will be said to be *exact* if both the functor

$$(\mathcal{C}, \mathcal{C}_\dagger) \rightarrow (\mathcal{D}, \mathcal{D}_\dagger)$$

of Waldhausen ∞ -categories and the functor

$$(\mathcal{C}, \mathcal{C}^\dagger) \rightarrow (\mathcal{D}, \mathcal{D}^\dagger)$$

of coWaldhausen ∞ -categories are exact.

We denote by $\mathbf{Fun}_{\mathbf{Exact}_\infty}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathbf{Fun}_{\mathbf{Trip}_\infty}^b(\mathcal{C}, \mathcal{D})$ spanned by the exact functors $\mathcal{C} \rightarrow \mathcal{D}$.

This definition, contrasted with the definition of exact functor of Waldhausen categories [3, Df. 2.4] appears to overburden the phrase “exact functor” slightly and create the possibility for some ambiguity; however, in Pr. 3.4 we will see that in fact no ambiguity obtains.

For now, let us construct the ∞ -category of exact ∞ -categories.

2.11. **Notation.** Denote by $\mathbf{Exact}_\infty^\Delta$ the following simplicial subcategory of $\mathbf{Trip}_\infty^\Delta$. The objects of $\mathbf{Exact}_\infty^\Delta$ are small exact ∞ -categories; for any two exact ∞ -categories \mathcal{C} and \mathcal{D} , let $\mathbf{Exact}_\infty^\Delta(\mathcal{C}, \mathcal{D})$ be the full simplicial subset of $\mathbf{Trip}_\infty^\Delta(\mathcal{C}, \mathcal{D})$ spanned by the exact functors. We write \mathbf{Exact}_∞ for the simplicial nerve of $\mathbf{Exact}_\infty^\Delta$.

2.12. **Proposition.** Denote by $w\mathbf{Exact}_\infty^0 \subset \mathbf{Exact}_\infty^0$ the subcategory consisting of those exact functors $\mathcal{C} \rightarrow \mathcal{D}$ whose underlying functor of ∞ -categories is a categorical equivalence that induce equivalences $h\mathcal{C}_\dagger \simeq h\mathcal{D}_\dagger$ and $h\mathcal{C}^\dagger \simeq h\mathcal{D}^\dagger$. The ∞ -category \mathbf{Exact}_∞ is the relative nerve of $(\mathbf{Exact}_\infty^0, w\mathbf{Exact}_\infty^0)$.

The formation of the opposite of a Waldhausen ∞ -category defines an equivalence

$$\mathbf{Wald}_\infty \xrightarrow{\sim} \mathbf{coWald}_\infty.$$

Since exact ∞ -categories are defined by fitting together the structure of a Waldhausen ∞ -category and a coWaldhausen ∞ -category in a self-dual manner, we obtain the following.

2.13. **Lemma.** The opposite involution on \mathbf{Trip}_∞ restricts to an autoequivalence

$$\mathrm{op}: \mathbf{Exact}_\infty \xrightarrow{\sim} \mathbf{Exact}_\infty.$$

This permits us to dualize virtually any assertion about exact ∞ -categories.

3. EXACT FUNCTORS BETWEEN EXACT ∞ -CATEGORIES

In this section we show that the inclusion $\mathbf{Exact}_\infty \hookrightarrow \mathbf{Wald}_\infty$ is fully faithful. For this, we use in a nontrivial way the additivity condition for exact ∞ -categories. In particular, this additivity actually guarantees a greater compatibility between pullbacks and pushouts and between fibrations and cofibrations than one might at first expect.

3.1. **Lemma.** *In an exact ∞ -category, a pushout square*

$$\begin{array}{ccc} X & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y', \end{array}$$

in which the morphism $X \twoheadrightarrow Y$ is ingressive is also a pullback square. Dually, a pullback square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y', \end{array}$$

in which the morphism $Y \rightarrow Y'$ is egressive is also a pushout square.

Proof. We prove the first statement; the second is dual. Since $X' \twoheadrightarrow Y'$ is ingressive, we may form the cofiber

$$\begin{array}{ccc} X' & \twoheadrightarrow & Y' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z', \end{array}$$

which is an ambigressive square. Hence the square

$$\begin{array}{ccc} X & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z', \end{array}$$

is also ambigressive, whence we conclude that

$$\begin{array}{ccc} X & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \twoheadrightarrow & Y', \end{array}$$

is a pullback square. □

The next pair of lemmas give a convenient way to replace pushout squares with exact sequences.

3.2. **Lemma.** *For any exact sequence*

$$X' \xrightarrow{i} X \xrightarrow{p} X''$$

of an exact ∞ -category C , the object W formed as the pushout

$$\begin{array}{ccc} X' & \twoheadrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & W \end{array}$$

is a direct sum $X \oplus X''$. Dually, the object V formed as the pullback

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X'' \end{array}$$

is a direct sum $X' \oplus X$.

Proof. We prove the first assertion; the second is dual. Choose a fibrant simplicial category D whose nerve is equivalent to C . Now for any object T , the shear map

$$\mathrm{Map}_D(X, T) \times \mathrm{Map}_D(X, T) \xrightarrow{\sim} \mathrm{Map}_D(X, T) \times \mathrm{Map}_D(X, T)$$

induces an equivalence

(3.2.1)

$$\begin{aligned} & (\mathrm{Map}_D(X, T) \times \mathrm{Map}_D(X, T)) \times_{\mathrm{id} \times i^*, (\mathrm{Map}_D(X, T) \times \mathrm{Map}_D(X', T)), \mathrm{id} \times 0}^h (\mathrm{Map}_D(X, T) \times \Delta^0) \\ & \quad \downarrow \sim \\ & (\mathrm{Map}_D(X, T) \times \mathrm{Map}_D(X, T)) \times_{i^* \times i^*, (\mathrm{Map}_D(X', T) \times \mathrm{Map}_D(X', T)), \Delta}^h \mathrm{Map}_D(X', T), \end{aligned}$$

where:

- i^* denotes the map $\mathrm{Map}_D(X, T) \rightarrow \mathrm{Map}_D(X', T)$ induced by $i: X' \rightarrow X$,
- 0 denotes a vertex $\Delta^0 \rightarrow \mathrm{Map}_D(X', T)$ corresponding to a zero map, and
- Δ denotes the diagonal map.

The source of (3.2.1) is the product of $\mathrm{Map}_D(X, T)$ with the space of squares of the form

$$\begin{array}{ccc} X' & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & T, \end{array}$$

in C , and the target is equivalent to the space of squares of the form

$$\begin{array}{ccc} X' & \xrightarrow{i} & X \\ i \downarrow & & \downarrow \\ X & \longrightarrow & T \end{array}$$

in C . Consequently, the map (3.2.1) specifies an equivalence

$$\mathrm{Map}_D(X, T) \times \mathrm{Map}_D(X'', T) \xrightarrow{\sim} \mathrm{Map}(W, T).$$

This equivalence is clearly functorial in T , so it specifies an equivalence $W \xrightarrow{\sim} X \oplus X''$. \square

3.3. Lemma. *In an exact ∞ -category, suppose that*

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ p \downarrow & & \downarrow q \\ X' & \xrightarrow{i'} & Y', \end{array}$$

is either a pushout square in which $X \twoheadrightarrow Y$ is ingressive or a pullback square in which $Y \twoheadrightarrow Y'$ is egressive. Then the morphism

$$\begin{pmatrix} -p \\ i \end{pmatrix}: X \twoheadrightarrow X' \oplus Y$$

is ingressive, the morphism

$$(i' \quad q): X' \oplus Y \twoheadrightarrow Y'$$

is egressive, and these maps fit into an exact sequence

$$X \twoheadrightarrow X' \oplus Y \twoheadrightarrow Y'.$$

Proof. We prove the assertion for pushout squares; the other assertion is dual. We form a diagram

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & V' & \longrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y' & \longrightarrow & Z \end{array}$$

in which every square is a pullback square. By the previous lemma, V' is a direct sum $X' \oplus Y'$, and V is a direct sum $X \oplus Y'$. The desired exact sequence is the top rectangle. \square

3.4. Proposition. *The following are equivalent for a functor $\psi: \mathcal{C} \rightarrow \mathcal{D}$ between two ∞ -categories with exact ∞ -category structures.*

- (3.4.1) *The functor ψ carries cofibrations to cofibrations, it carries fibrations to fibrations, and as a functor of exact ∞ -categories, ψ is exact.*
- (3.4.2) *The functor ψ carries cofibrations to cofibrations, and as a functor of Waldhausen ∞ -categories, ψ is exact.*
- (3.4.3) *The functor ψ carries fibrations to fibrations, and as a functor of coWaldhausen ∞ -categories, ψ is exact.*

Proof. It is clear that the first condition implies the other two. We shall show that the second implies the first; the proof that the third condition implies the first is dual. So suppose ψ preserves cofibrations and is exact as a functor of Waldhausen ∞ -categories. Because a morphism is egressive just in case it can be exhibited as a cofiber, ψ preserves fibrations as

well. A pullback square

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ p \downarrow & & \downarrow q \\ X' & \xrightarrow{i'} & Y' \end{array}$$

in which $p: X \rightarrow X'$ and $q: Y \rightarrow Y'$ are egressive can be factored as

$$\begin{array}{ccccc} X & \xrightarrow{\begin{pmatrix} -p \\ i \end{pmatrix}} & X' \oplus Y & \xrightarrow{\text{pr}_2} & Y \\ p \downarrow & & \downarrow \text{id} \oplus q & & \downarrow q \\ X' & \xrightarrow{\begin{pmatrix} -\text{id} \\ i' \end{pmatrix}} & X' \oplus Y' & \xrightarrow{\text{pr}_2} & Y' \end{array}$$

in which all three rectangles are pullbacks. By the previous lemma, the left hand square is an ambigressive pullback/pushout, so when we apply ψ , we obtain a diagram

$$\begin{array}{ccccc} \psi X & \xrightarrow{\begin{pmatrix} -\psi p \\ \psi i \end{pmatrix}} & \psi X' \oplus \psi Y & \xrightarrow{\text{pr}_2} & \psi Y \\ \psi p \downarrow & & \downarrow \text{id} \oplus \psi q & & \downarrow \psi q \\ \psi X' & \xrightarrow{\begin{pmatrix} -\text{id} \\ \psi i' \end{pmatrix}} & \psi X' \oplus \psi Y' & \xrightarrow{\text{pr}_2} & \psi Y' \end{array}$$

in which the right hand square is easily seen to be a pullback, and the left hand square, being an ambigressive pushout, is also an ambigressive pullback. \square

3.4.1. Corollary. *The forgetful functors $\mathbf{Exact}_\infty \rightarrow \mathbf{Wald}_\infty$ and $\mathbf{Exact}_\infty \rightarrow \mathbf{coWald}_\infty$ are fully faithful.*

In particular, we may say that a Waldhausen ∞ -category \mathcal{C} “is” an exact ∞ -category if it lies in the essential image of the forgetful functor $\mathbf{Exact}_\infty \rightarrow \mathbf{Wald}_\infty$, and we will treat this forgetful functor as if it were an inclusion. Since this functor is fully faithful, this is not a significant abuse of terminology. We make sense of the assertion that a coWaldhausen ∞ -category “is” an exact ∞ -category in a dual manner.

3.5. The essential image of the forgetful functor $\mathbf{Exact}_\infty \rightarrow \mathbf{Wald}_\infty$ is spanned by those Waldhausen ∞ -categories that satisfy the following three criteria.

- (3.5.1) The underlying ∞ -category is additive.
- (3.5.2) The class of morphisms that can be exhibited as the cofiber of some cofibration is closed under pullback.
- (3.5.3) Every cofibration is the fiber of its cofiber.

The essential image of the forgetful functor $\mathbf{Exact}_\infty \rightarrow \mathbf{coWald}_\infty$ is described in a dual manner.

4. THEORIES AND DUALITY

Recall from [3] the notion of a *theory* $\mathbf{Wald}_\infty \rightarrow \mathcal{E}_*$ for an ∞ -topos \mathcal{E} ; this is nothing more than a pointed functor that preserves filtered colimits. In light of the equivalence $\mathrm{op}: \mathbf{Wald}_\infty \xrightarrow{\sim} \mathbf{coWald}_\infty$, we may speak of a *theory* $\mathbf{coWald}_\infty \rightarrow \mathcal{E}_*$ as well, which is again nothing more than a pointed functor that preserves filtered colimits.

All of the results and constructions from [3] can be transferred to the setting of \mathbf{coWald} -hausen ∞ -categories by transport of structure. In particular, this permits us to speak of *virtual coWaldhausen ∞ -categories* and the *left derived functor* of theories $\mathbf{coWald}_\infty \rightarrow \mathcal{E}_*$.

4.1. Notation. Suppose \mathcal{E} an ∞ -topos. Denote by $\mathrm{Thy}^\vee(\mathcal{E})$ the full subcategory of the ∞ -category $\mathrm{Fun}(\mathbf{coWald}_\infty, \mathcal{E}_*)$ spanned by the \mathcal{E} -valued theories.

Of course there is a canonical equivalence between theories on Waldhausen ∞ -categories and theories on \mathbf{coWald} -hausen ∞ -categories.

4.2. Definition. Suppose \mathcal{E} an ∞ -topos. Then for any theory $\phi: \mathbf{Wald}_\infty \rightarrow \mathcal{E}_*$, the *dual theory* ϕ^\vee is the composite $\phi \circ \mathrm{op}: \mathbf{coWald}_\infty \rightarrow \mathcal{E}_*$. This construction clearly yields an equivalence of ∞ -categories $\mathrm{Thy}(\mathcal{E}) \xrightarrow{\sim} \mathrm{Thy}^\vee(\mathcal{E})$.

Using the dual theory construction, we transport the fundamental notion of additivity as well as the construction of additivizations.

4.3. Definition. Suppose \mathcal{E} an ∞ -topos. Then a theory $\mathbf{coWald}_\infty \rightarrow \mathcal{E}_*$ will be said to be *coadditive* just in case it is the dual theory of an additive theory $\mathbf{Wald}_\infty \rightarrow \mathcal{E}_*$. Write $\mathrm{Add}^\vee(\mathcal{E})$ for the full subcategory of $\mathrm{Thy}^\vee(\mathcal{E})$ spanned by the coadditive theories.

4.4. Notation. For any \mathbf{coWald} -hausen ∞ -category \mathcal{C} , write $\mathcal{F}^\vee(\mathcal{C})$ and $\mathcal{S}^\vee(\mathcal{C})$ for the \mathbf{coWald} -hausen cartesian fibrations

$$\mathcal{F}^\vee(\mathcal{C}) := \mathcal{F}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \rightarrow N\Delta \quad \text{and} \quad \mathcal{S}^\vee(\mathcal{C}) := \mathcal{S}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \rightarrow N\Delta.$$

In light of the equivalence $\mathrm{Thy}(\mathcal{E}) \simeq \mathrm{Thy}^\vee(\mathcal{E})$, we obtain the following basic results.

4.5. Lemma. *For any ∞ -topos \mathcal{E} , a theory $\psi: \mathbf{coWald}_\infty \rightarrow \mathcal{E}_*$ is coadditive just in case, for any integer $m \geq 0$, the exact functors $\mathcal{F}_m^\vee(\mathcal{C}) \rightarrow \mathcal{F}_{\{0\}}^\vee(\mathcal{C}) \simeq \mathcal{C}$ and $\mathcal{F}_m^\vee(\mathcal{C}) \rightarrow \mathcal{S}_m^\vee(\mathcal{C})$ induce an equivalence*

$$\psi(\mathcal{F}_m^\vee(\mathcal{C})) \xrightarrow{\sim} \psi(\mathcal{C}) \times \psi(\mathcal{S}_m^\vee(\mathcal{C})).$$

4.6. Lemma. *The inclusion $\mathrm{Add}^\vee(\mathcal{E}) \hookrightarrow \mathrm{Thy}^\vee(\mathcal{E})$ admits a left adjoint D^\vee , given by the formula*

$$D^\vee \psi \simeq \mathrm{colim}_n \Omega_\mathcal{E}^n \circ \Psi \circ \mathcal{S}^{\vee, n},$$

where Ψ is the left derived functor of ψ .

4.7. Definition. Let us call the left adjoint $D^\vee: \mathrm{Thy}^\vee(\mathcal{E}) \rightarrow \mathrm{Add}^\vee(\mathcal{E})$ the *Goodwillie coadditivization*.

4.8. Definition. Suppose \mathcal{E} an ∞ -topos. Then An *exact duality* on an \mathcal{E} -valued theory ϕ is an equivalence

$$\eta: \phi|_{\mathbf{Exact}_\infty} \xrightarrow{\sim} \phi^\vee|_{\mathbf{Exact}_\infty}$$

of the ∞ -category $\mathrm{Fun}(\mathbf{Exact}_\infty, \mathcal{E}_*)$.

4.9. Example. Suppose \mathcal{E} an ∞ -topos, and suppose $\rho: \mathbf{Cat}_\infty^* \rightarrow \mathcal{E}_*$ a pointed functor from the ∞ -category \mathbf{Cat}_∞^* of pointed ∞ -categories to \mathcal{E}_* that preserves filtered colimits. Then an equivalence $\rho \xrightarrow{\sim} \rho \circ \text{op}$ induces an exact duality on the composite

$$\mathbf{Wald}_\infty \rightarrow \mathbf{Cat}_\infty^* \rightarrow \mathcal{E}_*.$$

For instance, the interior functor $\iota: \mathbf{Cat}_\infty^* \rightarrow \mathbf{Kan}_*$ admits an equivalence $\iota \xrightarrow{\sim} \iota \circ \text{op}$; consequently, the theory $\iota: \mathbf{Wald}_\infty \rightarrow \mathbf{Kan}_*$ admits an exact duality.

The purpose of this section is to describe a circumstance in which an exact duality on a theory ϕ descends to an exact duality on its additivization $D\phi$, and to show that these conditions obtain when $\phi = \iota$, giving a functorial equivalence $K(\mathcal{C}) \simeq K(\mathcal{C}^{\text{op}})$ for exact ∞ -categories \mathcal{C} .

4.10. Note that $(D\phi)^\vee$ is by construction equivalent to $D^\vee \phi^\vee$, so such a result can be thought of as giving an equivalence between the additivization of ϕ and the coadditivization of ϕ^\vee .

Consequently, for an exact ∞ -category \mathcal{C} , we aim to produce a kind of duality between the cartesian fibration $\mathcal{S}^\vee(\mathcal{C}) \rightarrow N\Delta$ and the cocartesian fibration $\mathcal{S}(\mathcal{C}) \rightarrow N\Delta^{\text{op}}$.

More precisely, we will construct a functor $\tilde{\mathbf{S}}_*(\mathcal{C}): N\Delta^{\text{op}} \rightarrow \mathbf{Cat}_\infty^*$ such that $\tilde{\mathbf{S}}_*$ is a straightening of the cocartesian fibration $\mathcal{S}(\mathcal{C}) \rightarrow N\Delta^{\text{op}}$, and the composite of $\tilde{\mathbf{S}}_*$ with the functor $\text{op}: N\Delta^{\text{op}} \rightarrow N\Delta^{\text{op}}$ given by $[\mathbf{n}] \mapsto [\mathbf{n}]^{\text{op}}$ is a straightening of the cartesian fibration $\mathcal{S}^\vee(\mathcal{C}) \rightarrow N\Delta$.

In order to do this, we introduce thickened versions $\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}^\vee, \widetilde{\mathcal{S}}, \widetilde{\mathcal{S}}^\vee$ of the fibrations $\mathcal{F}, \mathcal{F}^\vee, \mathcal{S}, \mathcal{S}^\vee$.

4.11. Notation. Let $\widetilde{\mathbf{M}}$ be the following ordinary category. The objects are triples (m, i, j) consisting of integers $0 \leq i \leq j \leq m$, and a morphism $(n, k, \ell) \rightarrow (m, i, j)$ is a morphism $\phi: [\mathbf{m}] \rightarrow [\mathbf{n}]$ of Δ such that $k \leq \phi(i)$ and $\ell \leq \phi(j)$. Define a triple structure on the nerve $N\widetilde{\mathbf{M}}$ in the following manner. An edge $(n, k, \ell) \rightarrow (m, i, j)$ is ingressive if the underlying edge $\mathbf{m} \rightarrow \mathbf{n}$ of Δ is an isomorphism and if $\ell = \phi(j)$. Dually, an edge $(n, k, \ell) \rightarrow (m, i, j)$ is egressive if the underlying edge $\mathbf{m} \rightarrow \mathbf{n}$ of Δ is an isomorphism and if $i = \phi(k)$.

The fiber of the functor $N\widetilde{\mathbf{M}} \rightarrow N\Delta^{\text{op}}$ over a vertex $\mathbf{n} \in N\Delta^{\text{op}}$ is the ∞ -category $\mathcal{O}(\Delta^n)$. We regard this ∞ -category as equipped with the triple structure in which a square

$$\begin{array}{ccc} i & \longrightarrow & k \\ \downarrow & & \downarrow \\ j & \longrightarrow & \ell, \end{array}$$

regarded as a morphism $(i, j) \rightarrow (k, \ell)$, is ingressive if $j = \ell$, and it is egressive if $i = k$. Now one verifies easily that $(N\widetilde{\mathbf{M}}, N\widetilde{\mathbf{M}}_\dagger) \rightarrow (N\Delta^{\text{op}})^\flat$ is a pair cartesian fibration with fibers $(\mathcal{O}(\Delta^n), \mathcal{O}(\Delta^n)_\dagger)$, and $(N\widetilde{\mathbf{M}}, N\widetilde{\mathbf{M}}^\dagger)^{\text{op}} \rightarrow (N\Delta)^\flat$ is a pair cocartesian fibration with fibers $(\mathcal{O}(\Delta^n), \mathcal{O}(\Delta^n)^\dagger)^{\text{op}}$.

4.12. Construction. If \mathcal{C} is a Waldhausen ∞ -category, write $\widetilde{\mathcal{F}}(\mathcal{C})$ for the simplicial set over $N\Delta^{\text{op}}$ satisfying the following universal property. We require, for any simplicial set K and any map $\sigma: K \rightarrow N\Delta^{\text{op}}$, a bijection

$$\text{Mor}_{/(N\Delta^{\text{op}})}(K, \widetilde{\mathcal{F}}(\mathcal{C})) \cong \text{Mor}_{s\mathbf{Set}(2)}((K \times_{N\Delta^{\text{op}}} N\widetilde{\mathbf{M}}, K \times_{N\Delta^{\text{op}}} (N\widetilde{\mathbf{M}})_\dagger), (\mathcal{C}, \mathcal{C}_\dagger)),$$

functorial in σ . Here, the category $s\mathbf{Set}(2)$ is the ordinary category of pairs (X, A) of simplicial sets X equipped with a simplicial subset $A \subset X$. By [3, Pr. 3.9], the map $\widetilde{\mathcal{F}}(\mathcal{C}) \rightarrow N\Delta^{\text{op}}$ is a pair cocartesian fibration.

Dually, if \mathcal{C} is a coWaldhausen ∞ -category, write $\widetilde{\mathcal{F}}^\vee(\mathcal{C})$ for the simplicial set over $N\Delta$ satisfying the following universal property. We require, for any simplicial set K and any map $\sigma: K \rightarrow N\Delta$, a bijection

$$\text{Mor}_{/N\Delta}(K, \widetilde{\mathcal{F}}^\vee(\mathcal{C})) \cong \text{Mor}_{s\mathbf{Set}(2)}((K \times_{N\Delta} N\widetilde{\mathbf{M}}^{\text{op}}, K \times_{N\Delta} (N\widetilde{\mathbf{M}}^\dagger)^{\text{op}}), (\mathcal{C}, \mathcal{C}^\dagger)),$$

functorial in σ . By the dual of [3, Pr. 3.9], the map $\widetilde{\mathcal{F}}^\vee(\mathcal{C}) \rightarrow N\Delta$ is a pair cartesian fibration, and it is clear that

$$\widetilde{\mathcal{F}}^\vee(\mathcal{C}) \cong \widetilde{\mathcal{F}}(\mathcal{C}^{\text{op}})^{\text{op}}.$$

The objects of the ∞ -category $\widetilde{\mathcal{F}}(\mathcal{C})$ may be described as pairs (m, X) consisting of a nonnegative integer m and a functor of pairs $X: \mathcal{O}(\Delta^m) \rightarrow \mathcal{C}$. Dually, the objects of the ∞ -category $\widetilde{\mathcal{F}}^\vee(\mathcal{C})$ may be described as pairs (m, X) consisting of a nonnegative integer m and a functor of pairs $X: \mathcal{O}(\Delta^m)^{\text{op}} \rightarrow \mathcal{C}$.

Now if \mathcal{C} is a Waldhausen ∞ -category, we let $\widetilde{\mathcal{S}}(\mathcal{C}) \subset \widetilde{\mathcal{F}}(\mathcal{C})$ denote the full subcategory spanned by those pairs (m, X) such that for any integer $0 \leq i \leq m$, the object $X(i, i)$ is a zero object of \mathcal{C} , and for any integers $0 \leq i \leq k \leq j \leq \ell \leq m$, the square

$$\begin{array}{ccc} X(i, j) & \longrightarrow & X(i, \ell) \\ \downarrow & & \downarrow \\ X(k, j) & \longrightarrow & X(k, \ell) \end{array}$$

is a pushout. Dually, if \mathcal{C} is a coWaldhausen ∞ -category, let $\widetilde{\mathcal{S}}^\vee(\mathcal{C}) \subset \widetilde{\mathcal{F}}^\vee(\mathcal{C})$ denote the full subcategory spanned by those pairs (m, X) such that for any integer $0 \leq i \leq m$, the object $X(i, i)$ is a zero object of \mathcal{C} , and for any integers $0 \leq i \leq k \leq j \leq \ell \leq m$, the square

$$\begin{array}{ccc} X(k, \ell) & \longrightarrow & X(k, j) \\ \downarrow & & \downarrow \\ X(i, \ell) & \longrightarrow & X(i, j) \end{array}$$

is a pullback. Since ambigressive pullbacks and ambigressive pushouts coincide, we deduce that

$$\widetilde{\mathcal{S}}^\vee(\mathcal{C}) \cong \widetilde{\mathcal{S}}(\mathcal{C}^{\text{op}})^{\text{op}}.$$

4.13. Notation. As in [3], the constructions above yield functors

$$\widetilde{\mathcal{S}}: \mathbf{Wald}_\infty \rightarrow \mathbf{Cat}_{\infty, /N\Delta^{\text{op}}}^* \quad \text{and} \quad \widetilde{\mathcal{S}}^\vee: \mathbf{coWald}_\infty \rightarrow \mathbf{Cat}_{\infty, /N\Delta}^*.$$

The functor $\mathbf{M} \rightarrow \widetilde{\mathbf{M}}$ given by the assignment $(m, i) \mapsto (m, 0, i)$ induces a natural transformation $\widetilde{\mathcal{S}} \rightarrow \mathcal{S}$ over $N\Delta^{\text{op}}$ and a natural transformation $\widetilde{\mathcal{S}}^\vee \rightarrow \mathcal{S}^\vee$ over $N\Delta$.

In light of the uniqueness of limits and colimits in ∞ -categories [6, Pr. 1.2.12.9], one readily has the following.

4.14. **Proposition.** *If \mathcal{C} is a Waldhausen ∞ -category, then the functor $\widetilde{\mathcal{F}}(\mathcal{C}) \rightarrow N\Delta^{\text{op}}$ is a cocartesian fibration, and the marked map $\widetilde{\mathcal{F}}(\mathcal{C})^{\natural} \rightarrow \mathcal{S}(\mathcal{C})^{\natural}$ defined above is an equivalence of the cocartesian model structure over $N\Delta^{\text{op}}$. Dually, if \mathcal{C} is a coWaldhausen ∞ -category, then the functor $\widetilde{\mathcal{F}}^{\vee}(\mathcal{C}) \rightarrow N\Delta$ is a cocartesian fibration, and the marked map $\widetilde{\mathcal{F}}^{\vee}(\mathcal{C})^{\natural} \rightarrow \mathcal{S}^{\vee}(\mathcal{C})^{\natural}$ defined above is an equivalence of the cartesian model structure over $N\Delta$.*

4.15. In particular, we deduce that the constructions $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}^{\vee}$ yield functors

$$\widetilde{\mathcal{F}}: \mathbf{Wald}_{\infty} \rightarrow \mathbf{Wald}_{\infty, N\Delta^{\text{op}}}^{\text{cocart}} \quad \text{and} \quad \widetilde{\mathcal{F}}^{\vee}: \mathbf{coWald}_{\infty} \rightarrow \mathbf{coWald}_{\infty, N\Delta}^{\text{cart}}.$$

Note that for any exact ∞ -category and any integer $m \geq 0$, the pair structure on the fibers

$$\widetilde{\mathcal{F}}_m \mathcal{C} \subset \text{Fun}_{\mathbf{Pair}_{\infty}}^{\sharp}((\mathcal{O}(\Delta^m), \mathcal{O}(\Delta^m)_{\dagger}), (\mathcal{C}, \mathcal{C}_{\dagger}))$$

and

$$\widetilde{\mathcal{F}}_m^{\vee} \mathcal{C} \subset \text{Fun}_{\mathbf{Pair}_{\infty}}^{\sharp}((\mathcal{O}(\Delta^m), \mathcal{O}(\Delta^m)^{\dagger})^{\text{op}}, (\mathcal{C}, \mathcal{C}^{\dagger}))$$

is defined *objectwise*; that is, a morphism $X \rightarrow Y$ of $\widetilde{\mathcal{F}}_m \mathcal{C}$ (respectively, $\widetilde{\mathcal{F}}_p^{\vee} \mathcal{C}$) is ingressive (resp., egressive) just in case, for any integers $0 \leq i \leq j \leq m$, the morphism $X(i, j) \rightarrow Y(i, j)$ of \mathcal{C} is so.

4.16. We now pass to the functors that classify the fibrations $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}^{\vee}$, \mathcal{F} , \mathcal{F}^{\vee} to obtain functors

$$\widetilde{F}_*, \widetilde{\mathbf{S}}_*, : \Delta^{\text{op}} \times \mathbf{Wald}_{\infty}^0 \rightarrow \mathbf{Cat}_{\infty}^0 \quad \text{and} \quad \widetilde{F}_*^{\vee}, \widetilde{\mathbf{S}}_*^{\vee}: \Delta^{\text{op}} \times \mathbf{coWald}_{\infty}^0 \rightarrow \mathbf{Cat}_{\infty}^0.$$

The functor \widetilde{F}_* can be described as the composite

$$\Delta^{\text{op}} \times \mathbf{Wald}_{\infty}^0 \xrightarrow{\mathcal{O} \times U} \mathbf{Pair}_{\infty}^{0, \text{op}} \times \mathbf{Pair}_{\infty}^0 \xrightarrow{\text{Fun}_{\mathbf{Pair}_{\infty}}^{\sharp}} \mathbf{Cat}_{\infty}^0,$$

where \mathcal{O} denotes the functor

$$\mathbf{m} \mapsto (\mathcal{O}(\Delta^m), \mathcal{O}(\Delta^m)_{\dagger}),$$

and U denotes the forgetful functor

$$\mathcal{C} \mapsto (\mathcal{C}, \mathcal{C}_{\dagger}).$$

Dually, the functor \widetilde{F}_*^{\vee} can be described as the composite

$$\Delta^{\text{op}} \times \mathbf{coWald}_{\infty}^0 \xrightarrow{\mathcal{O}^{\text{op}} \times U^{\vee}} \mathbf{Pair}_{\infty}^{0, \text{op}} \times \mathbf{Pair}_{\infty}^0 \xrightarrow{\text{Fun}_{\mathbf{Pair}_{\infty}}^{\sharp}} \mathbf{Cat}_{\infty}^0,$$

where \mathcal{O}^{op} denotes the functor

$$\mathbf{m} \mapsto (\mathcal{O}(\Delta^m), \mathcal{O}(\Delta^m)^{\dagger})^{\text{op}},$$

and U^{\vee} denotes the forgetful functor

$$\mathcal{C} \mapsto (\mathcal{C}, \mathcal{C}^{\dagger}).$$

The functor $\widetilde{\mathbf{S}}_*$ (respectively, $\widetilde{\mathbf{S}}_*^{\vee}$) is then exhibited as the subfunctor of \widetilde{F}_* (resp., of \widetilde{F}_*^{\vee}) that to any pair $(\mathbf{m}, \mathcal{C})$ assigns the full subcategory

$$\widetilde{\mathbf{S}}_m(\mathcal{C}) \subset \text{Fun}_{\mathbf{Pair}_{\infty}}^{\sharp}((\mathcal{O}(\Delta^m), \mathcal{O}(\Delta^m)_{\dagger}), (\mathcal{C}, \mathcal{C}_{\dagger}))$$

(resp., the full subcategory

$$\tilde{\mathbf{S}}_m^\vee(\mathcal{C}) \subset \mathrm{Fun}_{\mathbf{Pair}_\infty}^\#((\mathcal{O}(\Delta^m), \mathcal{O}(\Delta^m)^\dagger)^{\mathrm{op}}, (\mathcal{C}, \mathcal{C}^\dagger))$$

spanned by those diagrams X such that for any integer $0 \leq i \leq m$, the object $X(i, i)$ is a zero object of \mathcal{C} , and for any integers $0 \leq i \leq k \leq j \leq \ell \leq m$, the square

$$\begin{array}{ccc} X(i, j) & \longrightarrow & X(i, \ell) \\ \downarrow & & \downarrow \\ X(k, j) & \longrightarrow & X(k, \ell) \end{array}$$

is an ambigressive pushout (resp., for any integers $0 \leq i \leq k \leq j \leq \ell \leq m$, the square

$$\begin{array}{ccc} X(k, \ell) & \longrightarrow & X(k, j) \\ \downarrow & & \downarrow \\ X(i, \ell) & \longrightarrow & X(i, j) \end{array}$$

is an ambigressive pullback).

Now suppose \mathcal{C} an exact ∞ -category. Since ambigressive pushouts and ambigressive pullbacks coincide in \mathcal{C} , it follows that there is a canonical equivalence $\tilde{\mathbf{S}}_* \circ \mathrm{op} \simeq \tilde{\mathbf{S}}_*^\vee$, where $\mathrm{op}: N\Delta^{\mathrm{op}} \xrightarrow{\sim} N\Delta^{\mathrm{op}}$ is the opposite automorphism of $N\Delta^{\mathrm{op}}$. We therefore deduce the following.

4.17. Theorem. *Suppose \mathcal{E} an ∞ -topos. For any pointed functor $\rho: \mathbf{Cat}_\infty^* \rightarrow \mathcal{E}_*$ that preserves filtered colimits, an equivalence $\rho \xrightarrow{\sim} \rho \circ \mathrm{op}$ induces a canonical exact duality on the Goodwillie additivization $D\rho$.*

Proof. The equivalence $\rho \xrightarrow{\sim} \rho \circ \mathrm{op}$, combined with the equivalence $\tilde{\mathbf{S}}_* \circ \mathrm{op} \simeq \tilde{\mathbf{S}}_*^\vee$, yields an equivalence

$$|\rho \circ \tilde{\mathbf{S}}_*| \simeq |\rho \circ \tilde{\mathbf{S}}_* \circ \mathrm{op}| \xrightarrow{\sim} |\rho \circ \mathrm{op} \circ \tilde{\mathbf{S}}_* \circ \mathrm{op}| \simeq |\rho^\vee \circ \tilde{\mathbf{S}}_*^\vee|. \quad \square$$

Applying this result to the functor ι yields the following.

4.17.1. Corollary. *Algebraic K -theory admits an exact duality,*

5. THEOREM OF THE HEART

In this section, we show that the Waldhausen K -theory of a stable, idempotent-complete ∞ -category with a bounded t-structure agrees with the K -theory of its heart. Amnon Neeman has provided an analogous result for his K -theory of triangulated categories [18]; given Neeman's result, our result here may be alternatively summarized as saying that the Waldhausen K -theory of a stable, idempotent-complete ∞ -category \mathcal{A} with a bounded t-structure agrees with Neeman's K -theory of the triangulated homotopy category $\mathcal{T} = h\mathcal{A}$ (denoted $K({}^w\mathcal{T})$ in [17]); this verifies a conjecture posed by Neeman [13, App. A] for such stable homotopy theories.

Our proof is quite straightforward. We combine Waldhausen's Localization Theorem with an Eilenberg swindle to prove that the Waldhausen K -theory of a stable, idempotent-complete ∞ -category with a bounded t-structure agrees with the K -theory of the subcategory of connective objects. Then we use duality to apply Waldhausen's Localization Theorem and an Eilenberg swindle again to show that the K -theory of the subcategory of connective objects agrees with the K -theory of the heart.

5.1. Notation. In this section, let C denote a small, idempotent complete stable ∞ -category equipped with a bounded t-structure. Let $E \subset C$ be a thick subcategory. Suppose that for any $X \in E$, both $\tau_{\leq 0}X \in E$ and $\tau_{\geq 0}X \in E$. Write

$$E_{\geq a} := E \cap A_{\geq a}, \quad E_{\leq b} := E \cap A_{\leq b}, \quad \text{and} \quad E^{\heartsuit} := E \cap A^{\heartsuit}.$$

5.2. Lemma. *The K -theory of the ∞ -category $E_{\leq -1}$, equipped with its maximal pair structure, vanishes.*

Proof. We apply [3, Cor. 8.2.1] to the ∞ -category $E_{\leq -1}$; this ensures that the functor

$$\Sigma_{E_{\leq -1}}^{\infty} : E_{\leq -1} \rightarrow \widetilde{\mathbf{Sp}}(E_{\leq -1})$$

induce an equivalence on K -theory when the left and right hand sides are endowed with their maximal pair structures. Note that the suspension functor on $E_{\leq -1}$ is the composite $\tau_{\leq -1} \circ \Sigma_C$. Since the t-structure is bounded, it therefore follows that $\Sigma_{E_{\leq -1}}^{\infty}$ is equivalent to the constant functor at 0. \square

5.3. Warning. Note that the lemma above applies only to the *maximal* pair structure on $E_{\leq -1}$, in which all morphisms are cofibrations. The K -theory of the “usual” pair structure on $E_{\leq -1}$, in which a morphism is ingressive just in case it induces a monomorphism on π_{-1} , turns out to agree with the K -theory of E .

In light of the Special Fibration Theorem [3, Cor. 10.8.4], we deduce the following.

5.4. Proposition. *The inclusion $E_{\geq 0} \hookrightarrow E$ induces an equivalence $K(E_{\geq 0}) \xrightarrow{\sim} K(E)$.*

Now we dualize this argument in the following manner. The opposite ∞ -category C^{op} is endowed with the dual t-structure, with $(C^{\text{op}})_{\leq -n} = (C_{\geq n})^{\text{op}}$. We first use Waldhausen's localization theorem to show that $K((E^{\text{op}})^{\heartsuit})$ can be exhibited as the fiber of the map

$$K((E^{\text{op}})_{\leq 0}) \rightarrow K((E^{\text{op}})_{\leq -1}^{\max}).$$

Then the very same argument employed to prove Lm. 5.2 applies to prove that $K((E^{\text{op}})_{\leq -1}^{\max})$ vanishes.

5.5. Lemma. *The functor $\tau_{\leq -1}$ induces a pullback square*

$$\begin{array}{ccc} K((E^{\text{op}})^{\heartsuit}) & \longrightarrow & K((E^{\text{op}})_{\leq 0}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K((E^{\text{op}})_{\leq -1}^{\max}), \end{array}$$

where the ∞ -categories that appear are equipped with the pair structure in which a morphism is ingressive just in case it induces a monomorphism on π_0 . (In particular, the pair structure on $(E^{\text{op}})_{\leq -1}$ is maximal.)

Proof. We consider the labeled Waldhausen ∞ -category $(E^{\text{op}})_{\leq 0}$ in which the labeled edges are those morphisms $X \rightarrow Y$ of $(E^{\text{op}})_{\leq 0}$ such that the induced morphism $\tau_{\leq -1}X \rightarrow \tau_{\leq -1}Y$ is an equivalence. Note that every labeled edge is ingressive, so *has enough cofibrations* in the sense of [3, Df. 9.27]. Thus we have a pullback square

$$\begin{array}{ccc} K((E^{\text{op}})^{\heartsuit}) & \longrightarrow & K((E^{\text{op}})_{\leq 0}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K(\mathcal{B}((E^{\text{op}})_{\leq 0}, w(E^{\text{op}})_{\leq 0})) \end{array}$$

by the Generic Fibration Theorem II [3, Th. 9.30]. One verifies immediately that the conditions of [3, Cor. 10.8.2] are satisfied, whence we obtain an identification

$$K(\mathcal{B}((E^{\text{op}})_{\leq 0}, w(E^{\text{op}})_{\leq 0})) \simeq K((E^{\text{op}})_{\leq -1}^{\text{max}}). \quad \square$$

Now we may employ Lm. 5.2 again, along with Cor. 4.17.1, to imply the following.

5.6. Theorem (Heart). *The inclusion $E^{\heartsuit} \hookrightarrow E$ induces an equivalence $K(E^{\heartsuit}) \simeq K(E)$.*

In light of Neeman's Theorem of the Heart, we obtain the following, which verifies some cases of a conjecture of Neeman [13, Conj. A.5].

5.6.1. Corollary. *For any idempotent-complete stable ∞ -category \mathcal{A} , if the triangulated homotopy category $\mathcal{T} = h\mathcal{A}$ admits a bounded t -structure, then we have canonical equivalences*

$$K(\mathcal{A}) \simeq K(\mathcal{T}^{\heartsuit}) \simeq K({}^w\mathcal{T}).$$

6. APPLICATION: ABELIAN MODELS FOR THE ALGEBRAIC G -THEORY OF SCHEMES

A trivial application of the Theorem of the Heart applies to give a new proof of the theorem of Gillet, Thomason, and Waldhausen [19, 1.11.7].

6.1. Proposition. *The algebraic K -theory of any Karoubian exact category E agrees with the algebraic K -theory of the ∞ -category $D^b(E)$ of bounded chain complexes in E .*

Proof. Embed E into the abelian category A of left exact functors $E^{\text{op}} \rightarrow \mathbf{Ab}$ via the Gabriel–Quillen embedding [19, 1.11.5, A.7.1, A.7.16], and apply the Theorem of the Heart to the inclusion $D^b(E) \subset D^b(A)$, with the standard t -structure on $D^b(A)$. \square

However, tilting theory provides other bounded t -structures on the ∞ -category $D^b(A)$. The K -theory of the heart of any of these t -structures will agree with the K -theory of A . Let us explore one class of examples now.

6.2. Example. Suppose X a noetherian scheme equipped with a dualizing complex $\omega_X \in D^b(\mathbf{Coh}(X))$. Then Arinkin and Bezrukavnikov [1], following Deligne, construct a family of t -structures on $D^b(\mathbf{Coh}(X))$ in the following manner. (Here we use *cohomological* grading conventions, to maintain compatibility with [1].) Write X^{top} for the underlying topological space of X , and define $\dim: X^{\text{top}} \rightarrow \mathbf{Z}$ a map such that $i_x^! \omega_X$ is concentrated in degree $-\dim(x)$. Suppose $p: X^{\text{top}} \rightarrow \mathbf{Z}$ a function — called a *perversity* — such that for any points $z, x \in X^{\text{top}}$ such that $z \in \overline{\{x\}}$, one has

$$p(x) \leq p(z) \leq p(x) + \dim(x) - \dim(z).$$

Let $D^{p \geq 0} \subset D^b(\mathbf{Coh}(X))$ be the full subcategory spanned by those complexes F such that for any point $x \in X^{\text{top}}$, one has $i_x^! F \in D^{\geq p(x)}(\mathcal{O}_{X,x})$; dually, let $D^{p \leq 0} \subset D^b(\mathbf{Coh}(X))$ be the full subcategory spanned by those complexes F such that for any point $x \in X^{\text{top}}$, one has $i_{x,*} F \in D^{\leq p(x)}(\mathcal{O}_{X,x})$. Then $(D^{p \geq 0}, D^{p \leq 0})$ define a bounded t-structure on $D^b(\mathbf{Coh}(X))$ [1, Th. 3.10].

The algebraic K -theory of the heart $D^{p \geq 0}$ of this t-structure now coincides with the G -theory of X . Let us list two special cases of this.

(6.2.1) Suppose S a set of prime numbers. Let \mathcal{E}_S be the full subcategory of $D^b(\mathbf{Coh}(\mathbf{Z}))$ generated under extensions by the objects

$$\mathbf{Z}, \quad \{\mathbf{Z}/p \mid p \in S\}, \quad \{\mathbf{Z}/p[1] \mid p \notin S\}.$$

Then \mathcal{E}_S is an abelian category whose K -theory coincides with the K -theory of \mathbf{Z} .

(6.2.2) For any noetherian scheme equipped with a dualizing complex $\omega_X \in D^b(\mathbf{Coh}(X))$, the K -theory of the abelian category of *Cohen–Macaulay complexes* (i.e., those complexes $F \in D^b(\mathbf{Coh}(X))$ such that the complex

$$\mathbf{D}F := \mathbf{R}\text{Mor}_{\mathcal{O}_X}(F, \omega_X)$$

is concentrated in degree 0, [20, §6]) agrees with the G -theory of X .

7. APPLICATION: A THEOREM OF BLUMBERG–MANDELL

In this section, we give a new proof of the theorem of Blumberg–Mandell [4] that establishes a localization sequence

$$K(\pi_0 E) \longrightarrow K(e) \longrightarrow K(E)$$

for any even periodic E_1 ring spectrum E with $\pi_0 E$ regular noetherian, where e denotes the connective cover of E . In light of [3, Pr. 13.16], the key point is the identification of the fiber term; this is the subject of this section.

7.1. Definition. Suppose Λ a connective E_1 ring. Then a left Λ -module M is said to be *coherent* if it is almost perfect and truncated. Write $\mathbf{Coh}_\Lambda^\ell \subset \mathbf{Mod}_\Lambda^\ell$ for the full subcategory spanned by the coherent modules, and write $G(\Lambda)$ for $G(\mathbf{Coh}_\Lambda^\ell)$.

7.2. Warning. In general, it is not the case that a perfect Λ -module is coherent; consequently, the usual Poincaré duality map $K \longrightarrow G$ for discrete rings does not have an obvious analogue for E_1 rings.

It turns out that G -theory is not a new invariant of E_1 rings, since we have the following new proof of the Dévissage Theorem of Blumberg–Mandell [4].

7.3. Proposition. *For any coherent E_1 ring Λ , the inclusion $N\mathbf{Mod}_{\pi_0 \Lambda}^{\ell, \text{fp}} \hookrightarrow \mathbf{Coh}_\Lambda^\ell$ of the nerve of the category of finitely presented $\pi_0 \Lambda$ -modules induces an equivalence*

$$G(\pi_0 \Lambda) \xrightarrow{\sim} G(\Lambda).$$

Proof. We note that $\mathbf{Coh}_\Lambda^\ell$ is the full subcategory of the ∞ -category of almost perfect Λ -modules spanned by those that are bounded for the t-structure given by [8, Pr. 8.2.5.18]. Furthermore, [8, Pr. 8.2.5.11(2)] applies to ensure that $\mathbf{Coh}_\Lambda^\ell$ is idempotent complete. Consequently, the Theorem of the Heart applies, and the proof is complete once one observes that the heart $\mathbf{Coh}_\Lambda^{\ell, \heartsuit}$ can be identified with $N\mathbf{Mod}_{\pi_0 \Lambda}^{\ell, \text{fp}}$ [8, Rk. 8.2.5.19]. \square

Consequently, from the point of view of topology, G -theory is a relatively coarse invariant.

Now we hope to compare the G -theory of an E_1 ring to the K -theory of the ∞ -category of truncated perfect modules. This requires a regularity hypothesis, which we formulate now.

7.4. Definition. Let us say that a coherent E_1 ring Λ is *almost regular* if any truncated perfect Λ -module has finite Tor dimension.

7.5. Example. If the graded ring $\pi_*\Lambda$ has finite Tor-dimension (e.g., if $\pi_*\Lambda$ is a regular noetherian ring), then the Tor spectral sequence ensures that Λ is almost regular.

The following result is now an immediate consequence of [8, Pr. 8.2.5.23(4)]

7.6. Proposition. *Suppose Λ a coherent E_1 ring that is almost regular. Then the ∞ -category $\mathbf{Perf}_\Lambda^{\ell,b}$ of perfect truncated left Λ -modules coincides with the ∞ -category $\mathbf{Coh}_\Lambda^\ell$ of coherent left Λ -modules.*

Assembling all this, we obtain the following formulation of the Localization Theorem of [4].

7.7. Theorem. *Suppose Λ a coherent E_1 ring spectrum that is almost regular, and suppose $S \subset \pi_*\Lambda$ a collection of homogeneous elements satisfying the left Ore condition such that a left Λ -module M is S -nilpotent just in case it is truncated. Then there is a fiber sequence of spaces*

$$G(\pi_0\Lambda) \longrightarrow K(\Lambda) \longrightarrow K(\Lambda[u^{-1}]),$$

7.8. Example. Here are some examples of fiber sequences resulting from this theorem.

(7.8.1) Consider the Adams summand L with its canonical E_∞ structure; its connective cover ℓ admits a canonical E_∞ as well [2]. The fiber sequence above becomes

$$K(\mathbf{Z}) \longrightarrow K(\ell) \longrightarrow K(L).$$

(7.8.2) Similarly, one can use the E_∞ structure on complex K -theory KU and on its connective cover to obtain

$$K(\mathbf{Z}) \longrightarrow K(ku) \longrightarrow K(KU).$$

(7.8.3) For any perfect field k of characteristic $p > 0$ and any formal group Γ of height n over k , consider the Lubin–Tate spectrum $E(k, \Gamma)$ with its canonical E_∞ structure and its connective cover $e(k, \Gamma)$ with its induced E_∞ structure [2]. In this case, the fiber sequence above becomes

$$K(\mathbf{W}(k)[[u_1, \dots, u_{n-1}]]) \longrightarrow K(e(k, \Gamma)) \longrightarrow K(E(k, \Gamma)).$$

(7.8.4) Given any E_1 structure on Morava K -theory $K(n)$ and a compatible one on its connective cover $k(n)$, one has a fiber sequence

$$K(\mathbf{F}_p) \longrightarrow K(k(n)) \longrightarrow K(K(n)).$$

8. APPLICATION: G -THEORY OF SPECTRAL DELIGNE–MUMFORD STACKS

The purpose of this final very brief section is simply to note that the G -theory of locally noetherian spectral Deligne–Mumford stacks is insensitive to derived structure.

8.1. Definition. For any spectral Deligne–Mumford stack \mathcal{X} , we write $\mathbf{Coh}(\mathcal{X})$ for the stable ∞ -category of coherent sheaves on \mathcal{X} [7, Df. 2.6.20], i.e., those quasicoherent sheaves that are almost perfect and locally truncated. Write $G(\mathcal{X})$ for the algebraic K -theory of $\mathbf{Coh}(\mathcal{X})$.

Now the Theorem of the Heart, combined with [7, Rk. 2.3.20], instantly yields the following.

8.2. Proposition. *For any locally noetherian spectral Deligne–Mumford stack \mathcal{X} with underlying ordinary Deligne–Mumford stack \mathcal{X}_0 , the embedding $N\mathbf{Coh}(\mathcal{X}_0) \hookrightarrow \mathbf{Coh}(\mathcal{X})$ induces an equivalence*

$$G(\mathcal{X}_0) \xrightarrow{\sim} G(\mathcal{X}).$$

Roughly speaking, just as G -theory is invariant under ordinary nilpotent thickenings, it turns out that G -theory is invariant under *derived* nilpotent thickenings as well.

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